HIGH SCHOOL STUDENTS' UNDERSTANDING OF THE FUNCTION CONCEPT[1]

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Introduction

In the United States and throughout the world, there is a cacophonous debate about the quality of mathematics education from pre-school through collegiate levels. We have local, national and international exams, many curriculum projects and a plethora of rhetoric about many issues. One of those issues concerns the mathematics education of students from under-represented and under-served populations. This means, generally, students at the lowest level of academic achievement and socio-economic status. It has been well documented that, while it is not the only factor, there is a strong correlation between socio-economic status and achievement in mathematics for elementary and secondary school students (Jordan, 2007 and Jordan and Levine, 2009). Anyon (1981) points out that even with a standardized curriculum, social stratification of knowledge along socio-economic lines takes place. There are also often references in the literature that algebra serves as a "gatekeeper" subject and that mathematics literacy is rapidly becoming a pre-requisite for economic access (Kamii, 1990, Moses et. al, 1989). In addition, there is evidence of a strong overlap between socio-economic status and students in the bottom quartile of mathematics achievement (National Mathematics Advisory Panel, 2008). At the same time there has been little research on the needs of students in these groups in relation to specific mathematics concepts (Lubienski and Bowen, 2000). The authors of this paper chose to focus on the students in the bottom quartile of mathematics achievement because of the general lack of attention given to this group, even in the conversation around providing mathematics for all.

For a while, the debate was about whether the education of students in this target population should focus on skills and procedural facility or on conceptual understanding. Then it seemed that people began to realize that both were essential, but still there are arguments about which should come first, which of the two potential kinds of learning depends on and is based on the other. There is considerable "data" that buttresses these arguments, but almost all of it comes from scores on mandated tests of various kinds. Unfortunately, although these tests may be effective in measuring, in some sense, skills and procedural knowledge, they tell us very little about conceptual understanding which is much harder to evaluate. Indeed, there is very little data about the conceptual understanding of students from the aforementioned target populations.

This paper reports the results of one example of an overall approach, the Algebra Project, designed to begin to fill that lacuna. We are not claiming that the results of this study are anything like a complete solution to the problems described in the previous paragraph. We are not even addressing the full range of difficulties. Rather, we are focusing on one very narrow part of the overall problem. The goals of the Algebra Project are: to help high school students from the target population in one country (the United States) to pass all state and federally mandated examinations; graduate from high school "on time"; do well enough on college admissions examinations to be accepted into college; and to have knowledge and understanding of high school mathematics sufficient to place into, and succeed, in credit-bearing college mathematics courses. We give a single example, an existence proof if you will, of one class that used a curriculum and pedagogy developed by Bob Moses' Algebra Project. As a result of this approach, most of the students developed sufficient procedural knowledge to achieve most of these goals. Moreover, using qualitative research techniques from Mathematics Education research, it was possible to see that they made a strong start on developing an understanding of one of the most important topics in high school mathematics --- the function concept. In particular, we will consider several difficulties that, according to the literature, high school and college students have in developing their understanding of the function concept and see what effect our curriculum and pedagogical strategy for functions had in helping our participants overcome these difficulties.

In Section 1 we review some of the literature on learning and teaching the concept of functions, focusing in particular on some specific conceptual difficulties that are reported. We will also refer to APOS Theory, which gives a description of a possible process by which the function concept can be learned. APOS Theory can be used, and in many studies has been used, successfully, as a strictly developmental perspective (e.g., Breidenbach et al. 1992), or as a strictly analytical evaluative tool (e.g., Dubinsky et al., to appear), or as both (e.g., Weller et al., 2011). In this study, it is used as a strictly analytical evaluation tool. APOS Theory focuses on models of what might be going on in the mind of an individual when he or she is trying to learn a mathematical concept and uses these models to evaluate student successes and failures in dealing with mathematical problem situations. We chose APOS Theory for this study because of its effectiveness in previous studies over the past 28 years (see Weller et al., 2003). In Section 2 we describe the Algebra Project, which is the national program under which this research was conducted. In Section 3 we give a detailed description of the study including the formal research question, the participants, the instructional treatment, and the research methodology. Our results will be presented in Section 4. Here we will consider five of the aspects of the function concept studied in the literature as reported in Section 1. Finally, in Section 5 we compare results on our participants' apparent understanding of aspects of the function concept as reported in Section 4 with that of other, often more mathematically sophisticated, participants as reported in the literature review in Section 1. In this last section, we will also draw some conclusions, discuss the limitations of the study, consider what might be some topics for future research in this area, and mention some possible Implications for teaching practice.

Section 1. Review of Literature

There is a vast literature on teaching and learning the function concept, stretching over at least the past 50 years. Two major themes run through this research: theoretical perspectives for analyzing what it means to understand the concept of function, and students' conceptual difficulties in understanding this concept. The contexts of both of these themes range from middle school to college students and include teachers, both pre-service and in-service. We treat each of these themes briefly in the next two sub-sections.

1.1 Theoretical perspectives on understanding the concept of function

The current study is an application of a theoretical perspective to the study of learning the function concept. Such applications have been a major part of mathematics education research for at least 45 years.

One of the earliest theoretical investigations of understanding the function concept was due to Piaget (Piaget et al., 1968/1977) who applied his theory of reflective abstraction to linear functions, proportions and relations. The APOS Theory approach to functions is strongly influenced by this work. After the appearance of Piaget's study of functions, several authors published various alternative theoretical perspectives. These include: Vinner and Hershkowitz (1980), Vinner and Dreyfus (1989), Sfard (1991), Sierpinska (1992), Thompson (1994), McGowen et al. (2000), Tall (2001), Akkoc & Tall (2002, 2003, 205), Balacheff (1995), Gaudin (2002), Mesa (2004), Jones (2006), Balacheff and Gaudin (2010), and Biehler (in press).

Finally, we point out that APOS Theory was introduced in Cottrill et al. (1996[2]). This theory focuses on an attempt to analyze the internal mental structures and mechanisms constructed and used by an individual as he or she is thinking about a mathematical concept. The model of thinking developed by this analysis is then related to the individual's apparent understanding of the concept.

1.2 Student conceptual difficulties in understanding the concept of function

It is possible to organize the content of much of the literature on conceptual difficulties with functions into categories such as: knowing what is and what is not a function; confusion of the definition of a function (univalence, the vertical line test) with the property of being one-to-one; representations of functions; functional notation; APOS stages of understanding the function concept (APOS stages are discussed later in this section); and composition of functions. Following is a discussion of most, if not all, of the major difficulties that have been studied, organized according to these categories.

Knowing what is, and what is not, a function

Many authors such as Sierpinska (1992), Carlson (1998) and Clement (2001) report that students, even those successful in College Algebra courses, think a function must be defined by a single algebraic formula. Moreover, in a study of junior high school teachers, Even (1990, p. 104) reports, "Seven (of ten) … subjects expressed the belief that all functions can be represented by using a formula." This tendency was also apparent from the limited variety of examples of functions given by students on written instruments and in interviews reported by Breidenbach et al., (1992).

An important issue in identifying a function is the univalence requirement: each element in the domain corresponds to exactly one element in the range. Even (1990) found that although many participants considered univalence to be important and appeared to understand what it means, almost none of them could explain *why* it is important and *why* functions are defined that way.

[2] The acronym APOS was introduced in 1996, but the ideas of the theory as they are understood today were formulated in the period 1984-1995.

Other difficulties include: an inability to recognize a function represented by finitely many discrete points on a Cartesian coordinate system (Markovitz, Z. et al., 1986); the belief that constant functions are not functions (Bakar & Tall, 1991); and the notion that a function must be defined by a single analytic formula, so that an expression with split domain represents two functions (Carlson, 1998, Vinner & Dreyfus, 1989).

In general, students are overwhelmed by the complexity of the function concept and do not have an overall grasp of it (Thompson, 1994, Akkoc &Tall, 2002, 2003, 2005).

Understanding the one-to-one property

Several studies show that students often require that the elements of two sets be in a one-to-one correspondence before they will agree that conditions have been met for a functional relationship (Markovitz et al., 1986, Thomas, 1975, Vinner, 1983, Leinhardt et al., 1990). This can lead to conceptual difficulties such as, "$f(x) = 12$ is often not considered a function, since it is not one-to-one" (Markovitz et al. 1986). In general, the literature reports considerable confusion between the univalence condition in the definition of function and the uniqueness condition in the one-to-one property. (See, for example, Even, 1990, Dubinsky & Harel, 1992).

Vertical line test

Most students coming from high school believe very strongly in the vertical line test as a definition of function (see, for example, Wilson, 1994), although in ways that can lead to conceptual difficulties, such as not being able to see that the graph of a parabola that is represented by the equation $x = y^2$ and opens horizontally can represent a function provided one considers y to represent an element of the domain and x an element of the range (Breidenbach et al., 1992).

Representations of functions

Much of the literature on functions focuses on multiple representations but in 1994, Thompson asserted that:

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of representation. Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of *anything* to students. (Thompson, 1994, p. 39).

He went on to introduce the idea of the "core concept of function" as the "something" that was being represented. His suggestion for this core concept was the connections we make among representational activities that produce a subjective sense of invariance. In a series of papers, Akkoc & Tall (2002, 2003, 2005) investigated the role of the core concept as a "fundamental building block of the mathematical curriculum".

According to research on representations of functions, the variety of representations understood by students and some teachers appears to be limited and many almost exclusively prefer representation by an algebraic expression. In particular, studies suggest that most participants think a table is not a function (Clement, 2001), students in College Algebra courses have difficulty constructing a function that represents a "real-world" situation (Carlson, 1998) and transitions between different representations such as tables, expressions, and graphs are problematic (Schwarz et al., 1990). Nyikahadzoyi (2006) found that prospective high school teachers in Zimbabwe could not distinguish functions from non-functions when they were represented by sets of ordered pairs. Indeed, many researchers concluded that the ordered pairs representation is too abstract for students in high school (Jones, 2006).

Even students who recognize several different types of representations of functions cannot form links of ideas across representations (Thompson, 1994, Demarois & Tall, 1999, Akkoc &Tall, 2002, 2003, 2005) and tend to identify the representation with that which is being represented (Thompson, 1984, Sierpinska, 1992).

Functional notation

There are many reports in the literature about student difficulties with functional notation. They tell us that, in general, many students are unfamiliar with the terminology that relates to the conceptual aspects of the mathematical notion of function (Vinner & Dreyfus, 1989). In particular, "students often do not understand the concept of variable and the $f(x)$ notation (e.g., they may not understand the distinction between $f(a)$ and finding the values of x for which $f(x) = a$)" (Dreyfus & Eisenberg, 1982). While the Algebra Project's Road Coloring unit does introduce the *f(x)* notation, algebraic representations of functions and functional notation are not heavily emphasized in the Algebra Project curriculum. Therefore functional notation and the more procedural knowledge of evaluating algebraic representations of functions for a given input were not heavily emphasized in this study.

APOS Stages

In the Algebra Project's pedagogical strategy, APOS stages (see below and Asiala et al., 1996 for a brief discussion of the Algebra Project and stages in APOS Theory), which are based on ideas of Piaget, are one way of evaluating a student's understanding of a mathematical concept and provide a road map for helping the student in that development. In this study, our main concern is with the *action stage*, which is the least powerful stage and the *process stage*, which is the next higher stage. An action stage of conception is described by Dubinsky and McDonald (2001) as a stage in learning where "a transformation of objects is perceived by the individual as essentially external and as requiring, either explicitly, or from memory, step-by-step instructions on how to perform the operation." A process stage occurs only after an action is repeated and the individual has the opportunity to reflect upon the action. Then "he or she can make an internal mental construction called a process, which the individual can think of as performing the same kind of action, but without the need of external stimuli."

In a study of 62 sophomore college students who were mathematics majors preparing to teach middle and high school mathematics, Breidenbach et al. (1992) found that, before an APOS-based instructional treatment, when asked to explain what is a function, only 25 appeared to have achieved an action stage and, of these, only 9 had progressed to a process stage. The remaining 37 gave responses that did not indicate any understanding of the concept at all. The students were asked to provide examples of functions and these examples indicated a similar distribution of APOS stages. These results tend to be confirmed by Carlson (1998) who found that students successful in College Algebra courses are mainly at the action stage with respect to the function concept. Nyikahadzoyi (2006) also found that most of her participants were at the action stage. Reed (2007) and Cerapio (2009) found similar results regarding the distribution of APOS stages.

Composition of functions

Lovell (1971) describes a study by Orton of understanding composition of functions by 48 participants from the upper half of ability range in a secondary school in the UK. His summary statement is "An understanding of the composition of functions was arrived at by a few of the abler participants in the fourth and fifth years". Consistent with these results is the report of Ayers et al. (1988) who studied 20 first year college students in the US who were taking a first course in Calculus and had just completed a separate, optional lab unit on composition of functions that used traditional, lecture-style pedagogy. The study found that on a test with 15 questions on the composition of functions, the success rate was about 45%. More recently, Jojo (2011) found that Calculus students in South Africa did not clearly understand composition and decomposition of functions.

Summary

This review of the literature has described research taking place over the 51-year period from 1960 to 2011. It is interesting to note that while there has been considerable work in developing various theoretical perspectives and using them to report on and analyze the myriad of difficulties that students at a variety of ages, ranging from high school to college and including students at the higher ability levels, have in developing a rich understanding of the function concept, less attention has been paid to applying the theoretical analyses to developing pedagogical strategies for helping students overcome these difficulties as is done, for example by Ayers et al, (1988), Breidenbach et al. (1992), McGowen et al. (2000), and Reed (2007). It is not surprising, therefore that although there has been some progress, the conceptual difficulties observed more than 50 years ago are, for the most part, still with us today.

In the remainder of this paper, we will focus on five of the categories of conceptual difficulties discussed above: knowing what is, and what is not a function; understanding the one-to-one property and the univalence condition in the definition of a function; representations of functions; APOS stages; and composition of functions. We did not include the vertical line test and functional notation in our study because, in our opinion, a focus on these topics may put too much emphasis on symbolic manipulation, as opposed to conceptual understanding, of functions. Our main concern will be to show how an APOS-based pedagogy can be used with a class of ill-prepared, very low-performing high school students from the lowest quartile of socio-economic levels, so as to bring them to an understanding of the function concept that it is comparable to that of students from the general population at the beginning of their college careers.

Section 2. The Algebra Project

The Algebra Project began in the 1980's when a small group of parents and teachers, working together with Bob Moses, an icon of the Civil Rights Movement, developed some pre-algebra classroom materials designed to develop algebraic thinking and provide an opportunity for middle school students in Cambridge, MA to gain access to algebra while still in middle school. The subject of algebra in particular was readily seen as a gatekeeper for access to college prep mathematics in high school for the young people of this community. Through grassroots organizing efforts this group was able to make a change in their community when in 1992 the Cambridge School District mandated that all middle school students be offered a full course in algebra (West & Davis, 2006). The Algebra Project grew out of this effort, with a small group of teachers and parents joining together to continue the work that they had started in developing curriculum and training teachers. At the same time, with the help of Bob Moses and his experience with grassroots organizing during the Civil Rights Movement of the 1960's, they focused their vision, as an organization, on the broader goal of making an impact on education policy and practice at local, state and national levels. During the period 1985-2000 the Algebra Project worked with students and teachers in over 15 cities across the country while providing professional development to over 700 teachers in low-income rural and urban schools nationwide (Moses & Cobb, 2001, West & Davis, 2006).

In 2001 The Algebra Project received an award from the National Science Foundation to support the development of four years of high school mathematics curricular material based on the Algebra Project pedagogical strategies. This curriculum was intended to target the lowest performing students in the nation's urban schools and would be designed to accelerate the learning of this group with the aim of having them ready for college level mathematics in four years from the 9th to 12th grade. The project's approach was to develop a curriculum for four years of high school mathematics, in which students form a cohort that stays together for all four years of high school, studying mathematics every day using the Algebra Project curriculum with teachers trained in Algebra Project pedagogy.

In order to address the large gaps in the mathematical background of the lowest performing students in our nation's schools the Algebra Project pedagogical methods incorporate the experiential learning model associated with John Dewey, Jean Piaget, Kurt Lewin and David Kolb. In this spirit the Algebra Project curriculum is designed to approach learning as an interactive process and to provide an environment in which students are allowed to experience and interact with the curriculum and take an active part in their own learning. According to Dewey, the philosophy of experiential learning also includes the idea of the "school itself as a social institution through which social reform can and should take place" (Dewey, 1916, 1938). Thus the Algebra Project sees the idea of the classroom as a place where social change ultimately begins. In this context, it is not farfetched to conjecture that the desired social change in the nation's poorest communities is intimately connected with the education of the youth of these communities.

The work of the Algebra Project is profoundly connected with the Civil Rights Movement because, "…with the ascendance of information technology and the increasing complexity of our society, mathematics joins reading and writing as a literacy needed for full citizenship. Like it or not, history has thrust mathematicians and specialists in mathematics education into the middle of a central American dilemma: the reconciliation of the ideals in the Declaration of Independence and the United States Constitution with the structures of race and caste and the legacies of slavery and Jim Crow" (Dubinsky & Moses, 2011).

The ideas of Willard Van Orman Quine have had a strong influence on the development of the Algebra Project curriculum and pedagogy. Quine was interested in how knowledge emerges from external experiences and how our theoretical language comes from a rigorous regimentation of the everyday language that we use to describe these experiences (Quine, 1990, 1992). As the students go through the process of mathematizing various events and experiences, the movement from ordinary discourse to a regimented language that mathematicians use is played out repeatedly in the Algebra Project curriculum.

The Algebra Project curriculum uses a process that incorporates the ideas about the regimentation of ordinary discourse from Quine and also closely reflects the Developmental Process of Piaget. The Algebra Project approach to experiential learning has two relatively unique features. First, students are not merely told about experiences, which may or may not be meaningful to them; they actually have the experiences. Second, we do not merely expect the mathematical content that we see in the experience to simply emerge and enter the students' minds. Instead, we have a definite pedagogical strategy for helping students construct in their minds the mathematics that is embodied in the experiences. Thus, each item of the curriculum begins by providing the students with a concrete experience. The students are then given the opportunity to observe, reflect, and write about the experience using their own voice in addition to drawing pictures to model the event. The next step is for students to formalize the language that they use to describe the event. And finally the students develop symbolic representations of the event. The Algebra Project curriculum developers believe that this process gives students who lack the necessary background the opportunity to use the experience as a frame of reference to draw from so that they can participate in meaningful mathematical discussions about the "mathematization" of the events they experience.

The unit of curriculum that we will focus on in this study is a version of the "Road Coloring Unit", originally developed by mathematician Greg Budzban of Southern Illinois University. The main topic addressed in the unit is the development of the function concept, including the definition of function, various representations of functions, as well as composition of functions and other function concepts that many students and even some teachers find difficult. One feature that contributes to making this unit of curriculum non-standard is that it is based on what was an open research problem in mathematics at the time that the unit was developed. The problem, known as the "Road Coloring Problem", can be stated as follows:

> Let G be a finite directed graph, where all the vertices have the same out-degree k. Let A be the alphabet containing the letters $1,\ldots,k$. A synchronizing coloring (also known as a collapsible coloring) in G is a labeling of the edges in G with letters from A such that (1) each vertex has exactly one outgoing edge with a given label and (2) for every vertex v in the graph, there exists a word w over A such that all paths in G corresponding to w terminate at v (Adler et al., 1977). The problem is to find a synchronizing coloring for any such G.

It was solved in 2009. (See Trahtman, A. N., 2009 and also Budzban & Feinsilver, 2011.)

"Another way to phrase this problem is to imagine that you live on a graph and your friend invites you to come over. When you ask for directions your friend gives you directions that will get you to his (sic) house no matter what vertex on the graph your house corresponds to" (Wikipedia, 2011). The experience for the students in this Algebra Project unit has to do with them using hula hoops and sticks to build representations of cities and connecting roads that satisfy various conditions and playing games with their structures. In this way, they construct finite directed graphs large enough to walk through in the classroom, looking for synchronizing instructions that solve the problem. The "mathematization" of this experience provides students with concrete representations of various function concepts including function composition. The Road Coloring unit has been piloted and tested in a number of 9th and 10th grade classrooms throughout the U.S. The current study focuses on a group of high school students that were in an Algebra Project classroom in the Southeastern U.S. that used a special version of the Road Coloring Unit as a part of their mathematics curriculum.

Section 3. Description of Study

3.1 Research Question

Following is the research question for this study:

> Using Algebra Project materials and pedagogy with a class of students from the lowest quartile of socioeconomic status, and academic performance who are generally underrepresented in education reform, what level of knowledge, ability and understanding regarding the concept of function can these students attain in their first three years of high school and how does that compare with what the literature tells us about college students, including pre-service teachers and in-service teachers?

3.2 Participants

The participants in the study reported in this paper were a cohort of 20 students in their Junior Year at a high school in south Florida referred to in Perry et al. (2010) by the pseudonym, Eagle High School. We chose this level because in many states, including Florida, high school is one place where students are expected to develop a strong foundation for understanding the function concept that is necessary in studying mathematics at the college level.

After observing the class conducted by the regular teacher during most of the students' 9th grade in school year 2006-2007, Algebra Project teachers took over teaching the class in March, 2007. The part of the Algebra Project curriculum that dealt with functions was done during school year 2008-2009. By that time, due to various reasons such as students leaving the neighborhood, the cohort was reduced to 15 students. These 15 are the participants for the study. The following description of the demographic background and academic ability of the participants is based on Wynne and Giles (2010).

At the beginning of the 10th grade, nine of the participants spoke both English and Kreyol; four spoke both English and Ebonics; and two spoke Kreyol with limited English proficiency. In addition, as an evidence of the participants' socio-economic status we note that all but two of the participants received free or reduced-fee lunch.

The Florida Comprehensive Achievement Test (FCAT) is an English and Mathematics state-mandated test administered in the third, eighth and tenth grades. Students take the test repeatedly until they attain a score of 3 on a scale of 1 to 5. In order to graduate, they must attain a score of 3, which is considered a proficiency score, on the tenth grade test. On the eighth grade test, 13 of our participants had a score of 1 and the other two had a score of 2. The students in the Algebra Project classroom were selected based on their (very low) performance in middle school and on their FCAT scores in the 8th grade.

Thus, it is fair to say that the 15 participants in this study were at the lowest level of socioeconomic status and academic achievement. And the school kept it that way. For example, when one student began to display strong performance in her other courses and in state-mandated exams, she was removed from the group. The results regarding the participants' knowledge and understanding of the function concept must be considered with this in mind and in the light of what the literature reports about students in general. For more information about their overall high school performance see Wynne and Giles (2010).

3.3 Instructional Treatment

As we indicated above, the development of the curriculum was heavily influenced by two theoretical perspectives, that of V. O. Quine and APOS Theory. We consider that what Quine identifies as a process for *mathematizing* events involves moving from ordinary discourse to a *regimented language*, that is, the language used in mathematics. Adapting his theories to the classroom, we developed a multi-step process for the students to mathematize their observations about an experience. The first step is for the students to describe their observations in their own words. We call these common sense representations *people-talk*. The next step is for them to focus attention on features in their people-talk sentences. These features correspond to the actions, processes and objects of APOS Theory and the students learn to recognize them in their sentences. This leads to the next step, which is to use the features to convert their people-talk statements into Quine's regimented representations which we call *feature-talk*. We then engage the students in the process of constructing *iconic* symbols for their features, that is, symbols that are pictorial and/or diagrammatic representations, and use them to express their feature-talk statements. Finally, they replace their iconic symbols in their feature-talk sentences with the abstract symbols and terminology used by mathematicians to obtain the mathematization of their feature-talk sentences.

The instructional treatment of functions took place during the Junior Year of the 15 students who are the participants of this study. All work was done in cooperative groups of 3 or 4 students, because we felt that teamwork would be a powerful aid for these participants in learning to function as students. This was preceded by a study of the integers. We did this for two reasons. First, these particular students, even after two years of high school, did not have much understanding of the integers, especially with respect to arithmetic operations involving negative numbers. Second, our initial focus in the study of functions was on functions whose domains and ranges were discrete sets, and in many cases this meant subsets of the set of integers.

The main pedagogical tool for applying the ideas of Quine and APOS Theory to helping the students develop their understanding of the integers is contained in a module called the Trip Line[3]. Following is a description of part of this Algebra Project unit, which covers the integers, and serves, as indicated above, to provide students with an experiential foundation for their study of functions.

[3] This module is available at: http://www.algebra.org/curriculum/unit/Trip-Line/

Consider the following two questions about MTA stations in Boston[4], one a "how many" and the other a "which way" question: "Where is Porter Square (*P*) in relation to Central Square (*C*) on the Red Line in Cambridge?" and "Where is Harvard Square (*H*) in relation to Kendall Square (*K*)?" Underlying both questions is the concept of the relative position of two stops on the Red Line. The answer to both questions is the same: two stops outbound, an answer to both "how many" and "which way". The geometrical representation of this answer is a displacement of two units outbound. Students learn to think of the movement from Central Square to Porter Square as starting at Central Square and moving two units outbound, and of the movement from Kendall to Harvard as starting at Kendall and moving two units outbound. Thus we have two movements, which have the same number of stops and are in the same direction. That is, these two movements represent the same displacement.

PLACE FIGURE 1 ABOUT HERE

The diagram presented in Figure 1 is an *iconic* representation of the students' observations about the trips. The people-talk representations are the statements: Porter Square is two stops outbound from Central Square. Harvard is two stops outbound from Kendall. Feature-talk involves explicit reference to location and relative positions of stops. This gives us addition as movement from the location of one stop to the location of another in one of two directions, and subtraction as the comparison of the location of the ending to the location of the starting stop. In other words, 'starting at the location of Kendall and moving two stops outbound one arrives at the location of Harvard' is feature-talk leading to addition, and 'the location of Harvard compared to the location of Kendall is 2 stops outbound' is feature-talk leading to subtraction. To obtain this mathematization, we select some stop as the benchmark. We then discuss with the students assigning symbols

[4] In the implementation that led to these questions, the students took a trip on the MTA. In other implementations they take trips around their city, their neighborhood or their school.

such as 0 for the benchmark, x_1 for the location of Kendall, x_2 for the location of Harvard, and Δx for the displacement. Then the first feature-talk sentence is mathematized into the equation,

$$x_1 + \Delta x = x_2,$$

and the second is mathematized into the equation

$$x_2 - x_1 = \Delta x.$$

(Dubinsky and Moses, 2011, pp. 304-306)

After taking their trip and writing their observations, the students construct a diagram, called a Trip Line, which is a straight line with various locations on the trip marked with points equidistant from each other. Making these markings, which represent integers, are actions in the sense of APOS Theory. Thinking about locations as movements from a benchmark to the location helps the students interiorize these actions into processes, and applying the operations of addition and subtraction to the locations leads the students to encapsulate the processes into mental objects which can be manipulated. After the students have worked with the experiences in which the integers and their operations are embodied, they learn the standard names of these processes and objects.

The study of functions, which followed this preliminary work on integers, used a version of the Algebra Project's Road Coloring[5] unit discussed above. Student work on the unit began with 7 school days during which the students used the materials to build physical representations of cities and connecting roads; played games using their cities; and held contests, all designed to form an experiential base for the function concepts which were to be studied. For example, Figures 2(a) and 2(b) represent cities that the students built. The circles were hula hoops, the numbers were written on slips of paper placed in the hula hoops and the paths from one circle to another were ribbons or sticks. The basic experience begins with one student standing in each hula hoop. Each student was then asked to move along the path from her or his circle to another circle. Notice that in Figure 2(a) the student in circle 3 does not know whether to move to circle 2 or circle 4. Discussing this phenomenon leads to the idea of univalence in the definition of function and the fact that Figure 2(a) does not represent a function. Figure 2(b) does and the fact that the students in circles 3 and 5 both arrive in circle 4 is the experiential basis for the concept

[5] This module is available at www.csupomona.edu/~robinwilson/Papers.html

of one-to-one. In a similar manner, the games and contests lead to experiences for function concepts such as inverse and composition.

PLACE FIGURE 2 ABOUT HERE.

Again, after the students have worked with the experiences in which functions and their properties are embodied, they learn the standard names of these processes and objects. This includes functional notation and the symbols for composition of functions.

We also used other representations of functions as arrow diagrams, sets of ordered pairs, discrete points on a graph, etc. We chose for our examples mainly functions whose domain and range are finite sets because we felt that such functions are best suited for experiences leading to the beginning of a conceptual understanding of the function concept. These experiences were followed by about six weeks of direct work on learning about functions. The pedagogical strategy here consisted of a series of dialogues about the experiences and the mathematics embedded in them. Each dialogue was followed by a worksheet. The characters of the dialogues were a mathematician and three students. In a somewhat Socratic spirit, the mathematician asked questions, the students made responses and all four engaged in discussions of the questions and answers. The students in the class took turns to play the roles of the mathematician and students by reading the dialogues. Most of the lines in the dialogues contained blanks for key words, which the reader had to fill in by understanding the context. If the student was unsuccessful, her or his group would help out and if that failed, the entire class tried together. The teacher was the arbiter[6].

Each dialogue ran for about 3-4 pages and was followed by a worksheet in which the students worked on problems. Some of these problems were about relating their experiential base to the mathematical concepts discussed in the dialogue and others were the usual problems that students would be expected to encounter in studying functions. There was no formal instruction on solving the problems, but as the students worked on them in their groups; the teacher was able to interact with them, giving help where necessary.

[6] The students were not anxious to read the dialogues and the teacher had to work hard to keep them at it. This turned out to be worth the trouble as one result of the project was an apparent significant improvement in their reading and comprehension skills.

The relations between the experiences with cities and function concepts led to the notion of one type of representation of functions called a “road subgraph”. This was a cross between a pictorial representation of the cities and roads the students had built and “played” with and a representation of a subgraph of a directed graph. The directed graph representation pervaded the entire curriculum.

Other representations were also emphasized. These included arrow diagrams (Figure 3), sets of ordered pairs, points on a Cartesian coordinate system (Figure 4), and algebraic expressions. Students were asked to recognize functions given by each type of representation, make their own examples, and, most important, transfer from one representation to another. Some, but not as much, emphasis was placed on representations of functions by tables, real world situations and curves without labeled points.

PLACE FIGURES 3, 4 ABOUT HERE

In addition to the issue of representation, which was present throughout the instructional treatment, other topics were also treated in a substantial manner in the context of each type of representation. These topics included: knowing what is, and what is not, a function; understanding the one-to-one property; and composition of functions. An important topic related to composition of functions concerned the equation $h = f \circ g$. Students were given examples of two of the three functions f, g, h and asked to find the third. These topics were chosen for emphasis because of their importance in understanding the function concept and because according to the mathematics education literature, they are problematic for college students including pre-service teachers (Breidenbach et al., 1992).

3.4 Research methodology

Immediately after they finished studying the Algebra Project material on functions, 14 of the 15 students completed a written instrument. This instrument contains 13 questions in which they were asked to: provide examples of a function; decide if given examples represented functions; evaluate functions given by various representations at specific points; evaluate the composition of two functions at given points; and solve several of the composition equations described above.

Three months later, 11 of the students agreed to an in-depth interview in which they were asked to: explain some of their responses on the written instrument; interpret the composition expressions , and apply their interpretations to specific examples; and solve additional composition equations using various representations.[7]

We analyzed the data from these two instruments and compared the results with the conclusions drawn from research over the past 40 years as described above in Section 1.

Section 4. Results

We organize our presentation of the results of the written instrument and interviews according to our focus on five categories. In Section 4.1 we present results on knowing what is, and what is not a function; in Section 4.2 we present results on understanding the one-to-one property and the univalence condition in the definition of a function; in Section 4.3 we present results on how students' knowledge and understanding of functions depended on the representations; in Section 4.4 we indicate the APOS stages that the students appeared to have reached; and in Section 4.5 we present results on their understanding of composition of functions.

In all five sections, the main source of data is the interviews. In Sections 4.1, 4.3, and 4.5, we also present data from the written instrument. A written instrument tends to give information about knowledge of facts and procedures whereas an interview can probe more deeply into the students' understanding of those facts and procedures. We used both sources because we felt that the combination of written instrument and interview can give deep and reliable information about the students' knowledge and understanding. In the case of Sections 4.4 and 4.6 we did not try to obtain any information from the written instruments. This is because, in the case of one-to-one and univalence, we were mainly interested in how the distinction between these two properties related to the students' understanding of functions, and in the case of APOS stages, students' knowledge of these stages was not part of their study of functions. The reliability of the scoring procedure for the data included in the tables was achieved by the two authors and two graduate students scoring the student responses independently and then negotiating differences.

[7] Full copies of both the written instrument and the interview protocol can be found on line at: www.csupomona.edu/~robinwilson/Papers.html

4.1 Knowing what is, what is not, a function.

The students were asked on the written instrument to give 3 examples of functions. An example was considered correct if it satisfied the main features of the definition of a function. That is, there must be evident, explicitly or implicitly, the presence of a domain set and a range set; there must be a rule to describe how to transform an element of the domain to an element of the range; and the function must satisfy the univalence condition which requires that each element in the domain of a function is transformed into one and only one element of the range. If it failed any of these criteria it was considered incorrect. On this task, which is a harder task than being asked to recognize given examples as functions or non-functions, we found that 64% successfully gave three correct examples, and 79% were successful at giving at least two correct examples of functions. Out of the 45 examples given, 35 were correct; thus 78% of the examples given were correct examples of functions. (See Tables 1a,b[8]). Following is a summary of the results of Tables 1a,b:

- nine of the 14 students that completed the written instrument, gave three correct examples, and one of them gave six correct examples,
- two gave two correct examples,
- one student gave one correct example, and
- two students gave no correct examples.

PLACE TABLES 1a,b ABOUT HERE

We next consider the students' ability to decide if an example could, or could not, be interpreted as a function and their ability, given an example of a function f and a specific element a of the domain, to determine the value of $f(a)$. We also considered how these abilities depended on the representation in which the function was given.

They did reasonably well with the 6 examples given to each of the 14 students on the written instrument. In 53 of the 84 (63%) examples, they were able to tell if it did or did not represent a function. Furthermore, it is possible that some of the 31 incorrect responses represented a misunderstanding of the question. For example, three students (who together gave 18 of the 31

[8] All student names are pseudonyms.

incorrect responses) did not answer the question, but rather simply explained (correctly for the most part) the process in the representation.

In the interview, we presented each student with six examples using representations emphasized in the Road Coloring unit (arrow diagrams, ordered pairs, directed graphs and points on a Cartesian coordinate system) and four examples using representations that were treated only lightly in the Road Coloring unit (real world situations and curves without labeled points). In addition we presented each student with three examples (tables and expressions) where, given a function and a specific domain element, the student was asked to evaluate the function at the domain point. These latter two representations were discussed in the Road Coloring unit. Overall, we recorded 143 student-example interactions, 66 of which referred to examples emphasized in the Road Coloring unit, 44 of which referred to non-Road Coloring emphasized examples, and 33 were given in the interview as a problem to evaluate a function at a point. If we exclude the examples related to real world situations and curves without labeled points, both of which were barely mentioned in the Road Coloring unit, then the results are that four of the 11 students were successful in recognizing that all of the examples presented to them were functions. Two others were able to correctly identify some of the examples from each representation as functions, and three others were able to recognize some of the examples from at least five different representations as functions. Regarding the two excluded representations, six of the 11 students were successful recognizing at least some of the real world situation examples as functions and one student was successful in recognizing the curve without points example.

PLACE TABLE 2 ABOUT HERE

For Table 2, we gave a ranking of 0-4 for each student interaction with each representation with 0 being the lowest and 4 being the highest. The students were very successful (an average score per student of better than 2.5) with the points on a Cartesian coordinate system, table and expression representations; moderately successful (better than 1.5 but not better than 2.5) with the arrow diagram, ordered pairs and directed graph representations; and did poorly (not better than 1.5) on the curve without labeled points and real world situation representations. It is worth noting that the students were successful with the table and expression representations of functions that they were exposed to in the Road Coloring Unit. Also, the students were the least successful with the representations involving a verbal description of a function.

4.2 Understanding the one-to-one property and the univalence condition in the definition of a function.

Analysis of the interview transcripts allowed us to provide an account of the students' understanding of: the one-to-one property; the requirement that a function be defined for every element of its domain; and the univalence requirement in the definition of a function. This analysis also provided information about their understanding of the distinction between the definition of a function and the one-to-one property. These topics were included in the version of the Road Coloring unit that the students used in the classroom.

Following is a summary of this analysis.

- nine students provided evidence suggesting that they understood that the definition of a function requires the function to be defined for every element of the domain,
- two students provided evidence of lack of understanding that the definition of function requires that the function be defined for each element of its domain,
- seven students provided evidence suggesting that they understood the univalence condition and its requirement in the definition of a function,
- four students provided evidence suggesting that they did not understand the univalence condition nor its role in the definition of function,
- nine students provided evidence indicating that they understood that a function does not have to be one-to-one.
- Two students provided evidence of a lack of understanding that a function does not have to be one-to-one.

Following is an excerpt from the interview of Camryn in response to question 2b in the written instrument, showing that she appears to understand that a function, for example one represented by an arrow diagram, must be defined for every element of its domain, it does not have to be one-to-one, and it must satisfy the univalence condition.

> Interviewer: Why? What makes it a function?
>
> Camryn: when every element in the domain has somewhere to go

Interviewer: ok, so is it a problem that two of the elements in the domain are both going to 4?

Camryn: no

Interviewer: that's not a problem?

Camryn: I don't think so

Interviewer: ok, how about that 2 in the domain is going to both 1 and 3 is that a problem

Camryn: yeah, that means it's not a function

Interviewer: ok, so is example b a function or not a function?

Camryn: not a function

Interviewer: why?

Camryn: because the domain repeats

In the following excerpt Bianca is discussing question 2b in the written instrument, and appears to be unclear about the distinction between the one-to-one property and the univalence condition and their role, or lack thereof in the definition of a function.

Interviewer: that's ok, that's ok… so then let's take a look at b. what do you think about b? Is this a function?

Bianca: yeah

Interviewer: why?

Bianca: because it's connecting, the numbers and it's showing you which direction to send the points to other numbers

Interviewer: so does it matter that I have 2 arrows pointing to 4?

Bianca: Not really, no.

Interviewer: and does it matter that I have 2 arrows coming from 2?

Bianca: no

4.3 Representations of functions

We obtained information about the extent to which our participants' understanding and knowledge of functions depended on the representations from three parts of the written instrument and interviews: examples of functions given by the student on the written instrument;

examples of functions given to the student on the written instrument and in the interview; and the questions in the interview about composition of functions.

On the written instrument, we asked each student to provide three examples of functions and we analyzed their responses in terms of the student's choice of representation. Of the 41 examples given by the students, 24 were arrow diagram representations of functions. Of these 24 representations, 20 were correct. Directed graphs were the second most frequently occurring examples with eight examples of which six were correct. There were seven examples of ordered pairs and two examples of points on a Cartesian coordinate system, all nine given correctly. (See Table 1b).

In the interviews, the students were asked to deal with several types of representations: arrow diagrams, sets of ordered pairs, directed graphs, points on a Cartesian coordinate system, curves without labeled points, tables, expressions, and analyses of "real-world" situations. For composition of functions the questions had to do with arrow diagrams, ordered pairs, points on a Cartesian coordinate system, or tables.

The rating scale of 0-4 in Table 2 rates each student's ability, as seen in her or his interview, to recognize with some understanding, functions and non-functions in each of eight representations. We see from this table that the students' ability to recognize what is and is not a function does not appear to be dependent on a single representation. In fact, five of the students were consistently successful in recognizing a function in six of the eight representations. In terms of individual representations the students received an average of 2.0 or higher for five representations (ordered pairs, directed graphs, points on a Cartesian coordinate system, tables and algebraic expressions). Students were also on average able to answer at least some of the questions in the arrow diagram and real world situation categories. The average score for arrow diagrams was 1.7 and the average score for real world situations was 1.3.

In addition to dealing with functions given in a variety of representations, there was also occasionally a comment indicating that the participant thought about solving a problem regarding a function given in one representation by transferring to a different, more convenient (for solving the problem) representation. Here is one example of Milca discussing question 4 on the written instrument. Milca is talking about evaluating a function represented by points on a Cartesian coordinate system at a value in its domain. He struggles to solve the problem in its original representation and then says that it would be easier for him if he had switched to an arrow diagram, thereby displaying flexibility in his thinking about representations of functions.

> Interviewer: ok, how about 4? This time I'm giving you the function as points on a Cartesian coordinate system. If you input 1 into the function, what would be your output?
> Milca: I'm not sure, 1? This one number?
> Interviewer: you're inputting 1, what would come out?
> Milca: I'm not sure… if you put 1, but don't you have to have something…
> Interviewer: well…
> Milca: the input is like the domain, but what about the range?
> Interviewer: well, can you find the range for me? If I tell you…
> Milca: oh, well 3!
> Interviewer: so the range or the output would be?
> Milca: 3
> Interviewer: 3, and how did you know that?
> Milca: cause as you told me the input was 1, I just had to find the output, which was 3.
> Interviewer: how did you know it was 3? How did you specifically find 3 as opposed to 1 or 2?
> Milca: if you think about it…. Well, I just knew, but if you had to draw an arrow diagram it would be a better way to show it.

4.4 APOS Stages

According to APOS Theory, a student who has little or no conception about a function is said to have a *pre-function* understanding. As the student learns about functions, the nature of her or his understanding of functions is described in three stages: *action*, *process*, and *object*. An APOS evaluation of an individual's understanding of function is described in terms of these stages. The lowest level of understanding is the action stage in which the learner requires very explicit instructions (such as a formula or algorithm) on how to transform a domain element to a range element and must do this explicitly one element at a time. With just such a conception, the individual is restricted to understanding functions only if they are given by expressions. After working with functions conceived as actions and reflecting on that work, the learner may make mental constructions regarding functions that perform exactly the same transformations as the actions, but do so entirely in her or his mind, without the need for explicit instructions. These mental constructions are the processes. A process conception is required for an individual to understand composition of functions, inverses of functions and other function properties (see Breidenbach et al. 1992). Finally, when an individual is able to perform actions or processes on functions, he or she is said to have an object conception of functions. Depending on the mathematical situation a student is working with, it may be appropriate to use an action conception (e.g., evaluating a function at a point) or a process conception (inverting a function), or an object conception (e.g., applying the operation of differentiation). A learner who can conceive of functions at all three stages is able to perform and understand various mathematical operations that require going back and forth between action and process conceptions (e.g., to form the composition of two functions) or between process and object conceptions (e.g., to perform arithmetic operations on functions). This close relation between mathematical concepts related to functions and the APOS stages and the effectiveness of APOS Theory as a tool for evaluating student understanding in many studies, including those involving functions (Breidenbach et al., 1992, Weller et al., 2003) is why we felt that APOS Theory is an appropriate measure of function understanding. In this study, we focus on action and process stages because they are the most important for studying functions at the Secondary level.

We analyzed the interview transcript of each participant to determine her or his APOS stage of function conception in relation to an individual function. We found that each of seven of the 11 participants interviewed gave at least 20 indications of having a process or action conception regarding a specific function with no more than six indications of a pre-function or inappropriate action conception. We would say that such participants have a process conception. Of the other four participants, they had some number (less than 20) of indications of a process or appropriate action conception, but as many or more of pre-function or inappropriate action indications. We would say that these might be in transition to a process conception, but there were still many indications of having a pre-function conception or being at the action stage. The full data is given in Table 3.

PLACE TABLE 3 ABOUT HERE.

Following is a typical example of a student who indicated a process conception of a function, represented by a table, when she referred to a transformation and an element of the domain going to an element of the range in question 5 of the written instrument.

> Interviewer: ok, let's take a look at number 5. This time I'm giving you the function as a table of values. If I tell you that we're inputting 3, what would be the output?
> Camryn: 2.
> Interviewer: how do you know that?
> Camryn: because I look on 3 and I got 2.
> Interviewer: so how do you know that those two pair together?
> Camryn: because the domain go… it transforms into the range, that's why they call it the range it's where it went, what they changed into.

4.5 Composition of function

In both the written instrument and the interviews, we asked students questions, some of which we considered to be difficult, about composition of functions. Our intention was to investigate the depth of their understanding of the function concept. We also felt that success in solving these problems was an indication of a process conception of function and in some cases, an indication of a process conception that was strong enough so that it could be reversed in the mind of a participant in order to solve a difficult composition problem (see Dubinsky & Harel, 1992).

Among the tasks that the students had to perform on the written instrument and in the interview, was to solve problems of the following types, which, as indicated in Section 3.3, were covered in the instruction:

1. Suppose f and g are two functions. Find the compositions $f \circ g$ and $g \circ f$.
2. Suppose $h = f \circ g$ is the composition of two functions f and g. Given h and g, find f.
3. Suppose $h = f \circ g$ is the composition of two functions f and g. Given h and f, find g.

Problems of the last two types are referred to in Table 5 as "hard" problems.

There were a total of eight such problems on the written instrument and 15 on the interview protocol. Following is a breakdown of the representations of the functions used:

- in eight of these problems both given functions were represented by arrow diagrams;
- in three problems both were represented by sets of ordered pairs;
- in three problems, both were represented by points on a Cartesian coordinate system;
- in five problems one function was represented by points on a Cartesian coordinate system and the other by a table;
- in two problems one function was represented by a set of ordered pairs and the other by an arrow diagram; and
- in two problems, both functions were represented by expressions.

In some of the above problems, the students were only asked to find individual values of the unknown function applied to specific elements of the domain. In others, they were asked to find a representation for the entire function.

There were three participants on the written instrument and two in the interview who were successful on most of the composition problems of the above three types. Other students were successful on some of the problems. The full details are in Tables 4 and 5.

PLACE TABLES 4 AND 5 ABOUT HERE

Those responses in Tables 4 and 5 that were correct give indication not only of a process conception but also of the ability of the participants to use their process conception in what, in some cases, amounts to mentally inverting the process of a function which allows them to solve some of these types of problems. For example, consider the following exchange with Michelle, looking for the function *f* given functions *f* o *g* and *g* represented by arrow diagrams in question 10 of the written instrument. Her thinking about domain numbers being transformed or moving to range numbers seems quite evident.

> Interviewer: Ok, so let's go on then to question 10. And this says that, like I said before I'm starting with the composition *f* composed with *g* and the function *g*, how could you find the function *f*?

> Michelle: Start with, start with the function *g*… and then go to *f* of *g* to make function *f*.
>
> Interviewer: ok
>
> Michelle: So it's gonna be 1 to 1, 1 to 3, so it's gonna be 1 to 3 for *f*, for the function *f*.
>
> Interviewer: ok
>
> Michelle: and then 2 to 3 and 3 to 2, so function *f* is gonna be 2, 2. Then it's gonna be 3 to 2 and 2 to 1 and it's gonna be 3 to 1 for function *f*.

Now consider the excerpt from Marco, looking for the function *g* given functions *f* o *g* and *f* represented by sets of ordered pairs in question 12 of the written instrument. Not only does he talk explicitly about the numbers moving, but his use of phrases such as: "we come over to 3 on function *f*, range 3, and we backtrack to the domain of 1" and "we go to range 1 on function *f* and we trace that back to domain 3", are strong indications that he is reversing the process of the function *f*. Moreover, the fact that he is working with functions represented by ordered pairs means that he is doing more than following arrows. The process of a function represented by ordered pairs does not reside in the representation; it must come from the participant.

> Interviewer: So let's try another one, number 12. This time I'm giving you the composition f circle g, the function f but we're asking you to find g
>
> Marco: Ok, so the function *f* would start at 1, 1 goes to 3 so we come over to 3 on function *f* range 3 and we backtrack to the domain of 1, so therefore 1 is just going to its own self, it goes to 1. Start at function 2 for *f* of *g*, to leads to 1. So we go to range 1 on function *f* and we trace that back to domain 3, so therefore 2 just lead to 3. Now we come to a function f of *g*, domain 3 which moves to range 2. So it go to range 2 of function *f* and take that back to the domain 2. So you have range 2 to function *f* so it's 3 goes to 2, so for function g it would be domain 1 range 1, domain 2 range 3, domain 3 range 2.

Section 5. Discussion and Conclusions

In this section, we discuss our conclusions and relate them to the Research Question posed in Section 3.1 and our theoretical framework; describe the limitations of the study; suggest some directions for future research; and point out some possible implications for teaching practice.

5.1 Research Question

It seems reasonable to conclude that the students in this study learned a great deal about the concept of function. This is indicated by the results above showing their ability to recognize and provide examples of functions; they did not appear to suffer from the misconception that a function has to be defined by an algebraic formula; their view of functions was not restricted to those represented by algebraic expressions as is the case with many college students and even some practicing teachers; they were able to deal successfully with a wide variety of representations and transfer from one representation to another; the majority of these students could successfully evaluate functions with various representations, identify functions when representations are presented to them and in some cases explain why a representation fails to be a function. It seems that these students began to develop an understanding of what is, and what is not, a function and can sometimes apply this understanding to unfamiliar situations. A large number of students in this study appeared to develop a process conception of function and this is consistent with the theoretical argument that suggests that such a development is critical to understanding the distinction between the univalence condition and the one-to-one property as well in the ability to work with the composition of functions.

Although only a small number of participants were successful with the hard composition problems, we want to pay special attention to the strong performance of three students (Camryn, Marco, Jermaine) on the hard composition problems appearing in the written instrument and the interview. These three students do not represent the "accident" of having three strong students in a population in which academic performance was generally low. As we indicated earlier, not only was this class originally selected so that all students were low performing, but during their entire high school careers, they were closely monitored by the school which even removed one student who began to rise above that level. The students who remained were apparently considered by the school to be somewhat hopeless and not capable of high performance. Our results suggest the opposite. The performance of our group as a whole on college-level problems about functions and the fact that three of the 11 did reasonably well on especially difficult problems suggests that there exists a pedagogical strategy (Section 3.3) that can lead to high performance by even some students about whom expectations are extremely low.

The students' ability to perform in this study did not drop from the sky. What they showed they understood about functions, they were explicitly taught and, impressively, they learned what they were taught. It is not surprising, for instance, that all of the examples of functions given by students were representations that they were familiar with from the Road Coloring unit, nor was it surprising that arrow diagrams and directed graphs were the most frequent examples because they were the representations the students encountered most frequently in the Road Coloring unit. On the other hand, it is interesting to note that all of the examples involving ordered pairs and points on a Cartesian coordinate system given by students were correct, even though these representations were not emphasized in the instruction.

In light of the above comments, it seems fair to say, regarding our research question (see Section 3.1), that most of the high school students in this study have at least begun the development of a level of knowledge, ability and understanding regarding the concept of function, that is comparable with college students. There are gaps as indicated above and these would need to be dealt with in the remaining time the students have in high school. If this were done, then we believe that these students would have received a quality education in the concept of function that would prepare them for dealing with functions in college.

All but one of the participants in our study graduated from high school on time and, at this writing, most are in, or about to enter college. This is not the place to discuss the, in our opinion still open, question of whether or not our society truly wishes ALL of its children to have a quality education overall. But if it does, then this study suggests that even for students at the lowest socio-economic status and academic achievement level, such an education, at least in mathematics, is possible.

5.2 Limitations of the study

One limitation of this study is that it focused only on understanding a function as a process with very little attention paid to an object conception of functions. This suffices for the operations in the curriculum at this level such as representations, inverses and composition. For the higher level mathematics they will encounter at the end of high school and in college, such as applying the operations of derivatives and integrals and solving differential equations, it will be necessary for them to also construct object conceptions of functions.

Another limitation is the small sample size of 15 students and the fact that, due to absences, the number of students included in the interviews was only 11.

In Section 1.1 we mentioned other theoretical perspectives that might be used, as we used APOS Theory, as a strictly analytical evaluative tool. We decided to use APOS Theory because of its success in other studies over the last 28 years (see Weller et al., 2003 and more recent studies) and because we felt that using a single perspective in a given study would give the work a greater focus. It may be that one can obtain better results than in this study using other perspectives and we hope that other researchers will investigate that possibility.

Finally, although we had informed consent of the students and their parents for this study, the school was reluctant to provide full information about the academic background of individual students. Therefore, our information about the students' academic background is extensive but not fully complete.

5.3 Directions for future research and implications for teaching

The most important direction for future studies would be to replicate this study with larger groups of students and to extend the pedagogical approach to include the construction of object conceptions of functions.

The main implication of this study for teaching practice is the realization that the students in the target population of the Algebra Project are capable of learning high school mathematics. They should not be shunted into remedial classes and forgotten. On the contrary, we feel that the results of the study reported here suggest that pedagogy can be developed that will be effective for students from the lowest quartile of the socio-economic scale and with the weakest academic preparation. We are convinced that these students should study the standard content of high school mathematics and the approach described in this paper may provide one way for them to succeed in that study.

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